

Spinor Fields on the Surface of Revolution and their Integrable Deformations via the mKdV-Hierarchy

Vadim V. Varlamov

Siberia State University of Industry, 654007 Novokuznetsk, Russia

Abstract

Spinor fields on surfaces of revolution conformally immersed into 3-dimensional space are considered in the framework of the spinor representations of surfaces. It is shown that a linear problem (a 2-dimensional Dirac equation) related with a modified Veselov-Novikov hierarchy in the case of the surface of revolution reduces to a well-known Zakharov-Shabat system. In the case of one-soliton solution an explicit form of the spinor fields is given by means of linear Bargmann potentials and is expressed via the Jost functions of the Zakharov-Shabat system. It is shown also that integrable deformations of the spinor fields on the surface of revolution are defined by a modified Korteweg-de Vries hierarchy.

1 Introduction

At present time methods of the theory of surfaces penetrate into many areas of theoretical and mathematical physics and become an important and inherent part of the modern physics. The deep relation between the theory of surfaces and soliton theory is well known, there exists a numerous literature devoted to this question (see, for example, [Sym85, Bob94]). Recently, Konopelchenko and Taimanov introduced a new approach related the methods of integrable systems with a theory of surfaces conformally immersed into 3- and 4-dimensional spaces [Kon96, KT95, KT96, Tai97a, Tai97b]. The main tool of this approach is a generalized Weierstrass representation for a conformal immersion of surfaces into \mathbb{R}^3 or \mathbb{R}^4 , and also a linear problem related with this representation. A consideration of the linear problem along with the Weierstrass representation allows to express integrable deformations of surfaces via such hierarchies of nonlinear differential equations as a modified Veselov-Novikov hierarchy, Davy-Stewartson hierarchy and so on. At present this approach intensively

used for the study of constant mean curvature surfaces, Willmore surfaces, surfaces of revolution and in many other problems related with differential geometry and physics [Kon96, Kon98, KL98a, KL98b, KT95, KT96, KL97, Mat97a, Mat97b, Mat98, Tai97a, Tai97b, Tai97c, Tai98, Var98a, Var98b, Yam99]. Further, it is known [Ken79, HO83, HO85, KL98b] that the Weierstrass representation is closely related with a Gauss map which widely used in string theory [BP88, Bud86, BDF86, BR87, VP94], since a string worldsheet may be considered as a surface conformally immersed into \mathbb{R}^4 . Thus, we have a very intriguing relation between string theory and soliton theory [CK96, KL98c, KL99].

On the other hand, there exists a natural formulation of the Weierstrass representation in terms of spinor representation of surfaces [At71, John80, Sul89, Abr89, KS96]. By virtue of this we have an interesting relation with spinor structures and spinor bundles, and also with a theory of the Dirac operator on spin manifolds [Bau81, LM89, Bär91, BFGK, Fr97a, Amm98]. In this direction, spinor representations of surfaces immersed into 3-dimensional spin manifold, and also upper bounds for the first eigenvalue of the Dirac operator on such surfaces have been considered in the recent papers [Fr97b, AF98, AAF98].

In the present paper we consider a relation between a Weierstrass representation in the Eisenhardt-Konopelchenko form [Eisen, Kon96] for a conformal immersion of surfaces into a 3-dimensional euclidean space and spinor representations of surfaces. The linear problem related with the mVN-hierarchy in this case is coincide with a 2-dimensional Dirac operator. The spinor field (ψ, φ) satisfying the 2-dimensional Dirac equation is understood as a smooth section of the spinor bundle over the Riemann surface. Integrable deformations of (ψ, φ) are defined by the mVN-hierarchy. In parallel with this we consider spinor representations of surfaces in terms of the theory of spin manifolds and establish an algebraic structure of Weyl half-spinors φ^+, φ^- of the 2-dimensional manifold (surface) immersed into a 3-dimensional spin manifold. It is shown that integrable deformations of the field (φ^+, φ^-) are also expressed via the mVN-hierarchy. Further, in section 4 we consider a surface of revolution and show that in this case the linear problem reduces to a well-known Zakharov-Shabat system [ZS71], the fundamental solutions (Jost functions) of which in the case of one-soliton solution are expressed via the linear Bargmann potentials [Bar49]. An explicit form of the spinor fields (ψ, φ) , (φ^+, φ^-) is given. In conclusion, using a well-known fact that in the case of the surface of revolution the mVN-hierarchy reduces to a modified Korteweg-de Vries (mKdV) hierarchy [KT95, Tai97a] we give an explicit dependence of the spinor fields on deformation variable t in the case of the one-soliton solution and the first member of the mKdV-hierarchy.

2 Spinor structures on manifolds and the Dirac operator on surfaces

Let us recall some basic facts about spinor structures on manifolds. Let $(M^{p,q}, g)$ be n -dimensional ($n = p + q$) oriented (time-oriented and space-oriented) pseudo-Riemann manifold and let $P \times_{SO(p,q)}$ be the bundle of all $SO(p, q)$ -frames of $(M^{p,q}, g)$. Then, a *spinor structure* of $(M^{p,q}, g)$ is a pair (Q, f) of a $\mathbf{Spin}(p, q)$ -principal bundle $Q \times_{\mathbf{Spin}(p,q)}$ and a continuous surjective map $f : Q \rightarrow P$ such that the diagram

$$\begin{array}{ccc} Q \times_{\mathbf{Spin}(p,q)} \times \mathbf{Spin}(p, q) & \rightarrow & Q \times_{\mathbf{Spin}(p,q)} \\ \downarrow f \times \lambda & & \downarrow f \\ P \times_{SO(p,q)} \times SO(p, q) & \longrightarrow & P \times_{SO(p,q)} \end{array} \quad \begin{array}{c} \searrow \\ \nearrow \end{array} M$$

commutes, here λ is a double covering $\mathbf{Spin}(p, q) \rightarrow SO(p, q)$.

Further, an oriented pseudo-Riemann manifold with a spinor structure (Q, f) is called a *spin manifold*. The complex vector bundle

$$S = Q \times_{\mathbf{Spin}(p,q)} \Delta_{p,q}$$

associated with the $\mathbf{Spin}(p, q)$ -principal bundle $Q \times_{\mathbf{Spin}(p,q)}$ by means of the spinor representation $\Delta_{p,q}$ is called a *spinor bundle* of the manifold $(M^{p,q}, g)$. A smooth section $\varphi \in \Gamma(S)$ of the bundle S is called a *spinor field* on the manifold $(M^{p,q}, g)$. In case of even dimension, $p - q \equiv 0, 2, 4, 6 \pmod{8}$, the spinor bundle S splits into two subbundles $S = S^+ \oplus S^-$, where

$$S^\pm = Q \times_{\mathbf{Spin}(p,q)} \Delta_{p,q}^\pm,$$

here $\Delta_{p,q}^\pm$ are the spaces of half-spinors (Weyl spinors).

Let us consider now a 3-dimensional oriented Riemann manifold $M^{3,0}$ with a fixed spin structure and also a surface $M^{2,0}$ conformally immersed into $M^{3,0}$. The surface $M^{2,0}$ is considered as a space-oriented Riemann manifold with a spin structure. Moreover, since the normal bundle of $M^{2,0}$ is trivial, then the spin structure of $M^{3,0}$ induces a spin structure on the surface $M^{2,0}$. Let Φ be a spinor field on the manifold $M^{3,0}$. It is obvious that the field Φ is a section of 2-dimensional spinor bundle $S = Q \times_{\mathbf{Spin}(3,0)} \Delta_{3,0}$, since $\mathbf{Spin}(3, 0) \cong \mathbf{Spin}(0, 3)$ and $\mathcal{C}_{0,3}^+ \cong \mathbb{H}$, then $\dim \Delta_{3,0} = 2$ (here \mathbb{H} is a quaternion algebra, $\mathcal{C}_{p,q}^+$ is a Clifford algebra of all even elements). On the other hand, $\mathbf{Spin}(3, 0) \cong Sp(1) \cong SU(2) \cong S^3$ [Port69] and $\mathcal{C}_{3,0} \cong \mathbb{C}_2 \cong M_2(\mathbb{C})$. Further, it takes to find a restriction of the spinor bundle $S = Q \times_{\mathbf{Spin}(3,0)} \Delta_{3,0}$ of the manifold $M^{3,0}$ onto a spinor bundle $S_{M^{2,0}} = Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}$ of the surface $M^{2,0}$ conformally immersed into $M^{3,0}$. Let ϕ is a spinor field on the surface $M^{2,0}$. Obviously, this field is a section of

2-dimensional spinor bundle $S = Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}$, since $\mathbf{Spin}(2,0) \cong \mathbf{Spin}(0,2)$ and $\mathcal{C}_{0,2}^+ \cong \mathcal{C}_{0,1} \cong \mathbb{C}$, then $\mathbf{Spin}(2,0) \cong U(1) \cong S^1$. Moreover, the spinor bundle of the surface $M^{2,0}$ splits into two subbundles,

$$S = S^+ \oplus S^-,$$

where $S^\pm = Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}^\pm$. Respectively, a smooth section $\phi \in \Gamma(S)$ of the bundle S has a form $\phi = \phi^+ + \phi^-$, where [Fr97b]

$$\phi^+ = \frac{1}{2}(\phi + i\vec{N}\phi), \quad \phi^- = \frac{1}{2}(\phi - i\vec{N}\phi), \quad (1)$$

here $\vec{N} = \mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2$.

Remark. Let us consider in more details algebraic definition of the spinors (1). At first, for the 3-dimensional Clifford algebra $\mathcal{C}_{3,0}$ associated with the manifold $M^{3,0}$ there exists an isomorphism $\mathcal{C}_{3,0} \cong \mathbb{C}_2$, where \mathbb{C}_2 is an algebra of complex quaternions. Further, according to [Che54, Lou81] spinors are considered as elements of the minimal left ideal $I_{p,q} = \mathcal{C}_{p,q}e_{pq}$, where e_{pq} is a primitive idempotent $e_{pq} = \frac{1}{2}(1 + \mathbf{e}_{\alpha_1}) \dots \frac{1}{2}(1 + \mathbf{e}_{\alpha_k})$, $k = q - r_{q-p}$, r_{q-p} are the Radon-Hurwitz numbers defined by the recurrence formula $r_{i+8} = r_i + 4$ and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

The primitive idempotent of $\mathcal{C}_{3,0} \cong \mathbb{C}_2$ has a form $e_{30} = \frac{1}{2}(1 + \mathbf{e}_0) \sim \frac{1}{2}(1 + i\mathbf{e}_{12})$, since in this case $k = q - r_{q-p} = 0 - r_{-2} = 0 - (r_6 - 4) = 1$. It is obvious that $I_{3,0} = \mathcal{C}_{3,0}e_{30} \cong \mathbb{C}_2e_{30} \cong \mathbf{M}_2(\mathbb{C})e_{30}$, therefore for the algebraic definition of the spinors (1) we have

$$\begin{aligned} \varphi^+ &= e_{30}^+ I_{3,0}, \\ \varphi^- &= e_{30}^- I_{3,0}, \end{aligned}$$

here

$$e_{30}^+ = \frac{1}{2}(1 + i\mathbf{e}_{12}), \quad e_{30}^- = \frac{1}{2}(1 - i\mathbf{e}_{12})$$

are the mutually orthogonal idempotents.

Further, according to [Fr97b] suppose that the spinor Φ on the manifold $M^{3,0}$ is a real Killing spinor, i.e. there exists a number $\lambda \in \mathbb{R}$ such that for any tangent vector $\vec{T} \in \tau_{M^{3,0}}$ the derivative of the spinor Φ in the direction of \vec{T} is given by the equation

$$\nabla_{\vec{T}}^{M^{3,0}} \Phi = \lambda \cdot \vec{T} \cdot \Phi.$$

Then the Dirac equation for the spinors (1) may be written as follows [Fr97b]

$$\begin{aligned} D\varphi^- &= (-2\lambda + iH)\varphi^+, \\ D\varphi^+ &= (-2\lambda - iH)\varphi^-, \end{aligned}$$

where H is a mean curvature of the surface $M^{2,0}$ and

$$D\varphi^\pm = \mathbf{e}_1 \cdot \nabla_{\mathbf{e}_1}^{M^{2,0}} \varphi^\pm + \mathbf{e}_2 \cdot \nabla_{\mathbf{e}_2}^{M^{2,0}} \varphi^\pm$$

is a Dirac operator of $M^{2,0}$. On the other hand, if Φ is a parallel spinor on the manifold $M^{3,0}$ ($\lambda = 0$), then

$$\begin{aligned} D\varphi^- &= iH\varphi^+, \\ D\varphi^+ &= -iH\varphi^-. \end{aligned} \tag{2}$$

3 The spinor and Weierstrass representations of surfaces and the modified Veselov-Novikov hierarchy

Let S be a connected Riemann surface with a local complex coordinate z and let $P \times_G$ be a principal bundle of S with the structure group G (G is a group of fractional linear transformations). Then the *spinor representation of a surface* is given by the following diagram

$$\begin{array}{ccc} Q \times_{\tilde{G}} & \xrightarrow{\chi} & Q \times_{\mathbf{Spin}(2,0)} \\ \mu \downarrow & & \downarrow f \\ P \times_G & \xrightarrow{\omega} & P \times_{SO(2,0)} \\ \downarrow & & \downarrow \\ S & \xrightarrow{g} & M^{2,0} \end{array}$$

Here $g : S \longrightarrow \mathbb{C} \cup \{\infty\}$ is a Gauss map, which can be viewed as an identification (via the stereographic projection) of the surface $M^{2,0}$ and $\mathbb{C} \cup \{\infty\}$, ω is a bundle map of $P \times_G$ into $P \times_{SO(2,0)}$ satisfying the integrability condition $\text{Re } d\omega = 0$ [KS96]. \tilde{G} is a double covering of G , i.e. there is a mapping $\lambda : \tilde{G} \longrightarrow G$. According to [Roz55] the group \tilde{G} is a group of complex matrices of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$, and $\text{Ker } \lambda$ consists of the two matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Further, if (ω, g) is a bundle map of $P \times_G$ into $P \times_{SO(2,0)}$, then (i) there is a unique structure $Q \times_{\tilde{G}}$ on S such that ω lifts to a bundle map $\chi : Q \times_{\tilde{G}} \longrightarrow Q \times_{\mathbf{Spin}(2,0)}$; (ii) there are exactly two such lifts ψ , and these differ only by sign.

On the other hand, the Weierstrass representation [Weier] describes a conformal immersion of minimal surfaces into 3-dimensional Euclidean space:

$$f = \text{Re} \left[\int (1 - s^2, i(1 + s^2), 2s)\mu(s) \right] : M^{2,0} \longrightarrow \mathbb{R}^{3,0},$$

where

$$s = \frac{H(w)}{G(w)}, \quad \mu(s) = G^2 \frac{dW}{ds},$$

and $H(w), G(w)$ are holomorphic functions defined in a circle or in all complex plane. The Weierstrass and spinor representations are related by the equation [KS96]

$$f = \sigma(\psi) = (\varphi_1^2 - \varphi_2^2, i(\varphi_1^2 + \varphi_2^2), 2\varphi_1\varphi_2),$$

where $\psi = (\varphi_1, \varphi_2)$ is a pair of sections of the bundle $Q \times_{\tilde{G}}$, and

$$\mu(s) = \varphi_1^2, \quad s = \varphi_2/\varphi_1. \quad (3)$$

In connection with this let us consider a Weierstrass representation in the Eisenhardt-Konopelchenko form [Eisen, Kon96]:

$$\begin{aligned} X^1 + iX^2 &= i \int_{\Gamma} (\bar{\psi}^2 dz' - \bar{\varphi}^2 d\bar{z}'), \\ X^1 - iX^2 &= i \int_{\Gamma} (\varphi^2 dz' - \psi^2 d\bar{z}'), \\ X^3 &= - \int_{\Gamma} (\psi \bar{\varphi} dz' + \varphi \bar{\psi} d\bar{z}'), \end{aligned} \quad (4)$$

where Γ is an arbitrary curve in the complex plane \mathbb{C} , ψ and ϕ are complex-valued functions on variables $z, \bar{z} \in \mathbb{C}$ satisfying the following linear system (2-dimensional Dirac equation):

$$\begin{aligned} \psi_z &= p\varphi, \\ \varphi_{\bar{z}} &= -p\psi, \end{aligned} \quad (5)$$

where $p(z, \bar{z})$ is a real-valued function, and $p = \frac{u}{2}H$, where H is a mean curvature of the surface, $u = |\psi|^2 + |\varphi|^2$. If to interpret the functions $X^i(z, \bar{z})$ as coordinates in the space $\mathbb{R}^{3,0}$, then the formulae (4),(5) are defined a conformal immersion of the surface $M^{2,0}$ into $\mathbb{R}^{3,0}$. The formulae (4) reduce to a classical Weierstrass representation if suppose

$$\varphi = G, \quad \psi = H.$$

Further, it is well-known that the mVN-hierarchy is related with the system (5), see [Kon96, KT95, KT96, Tai97a]. This hierarchy is appear as compatibility conditions

$$\left[\frac{\partial}{\partial t_n} - A_n, L \right] = B_n L, \quad (6)$$

where the Lax operator L and deformation operators A_n, B_n are defined by the following expressions

$$\begin{aligned} L &= \begin{pmatrix} \partial & -p \\ p & \bar{\partial} \end{pmatrix} \\ A_n &= \partial^{2n+1} + \sum_{i=0}^{2n-1} X^i \partial^i + \bar{\partial}^{2n+1} + \sum_{i=0}^{2n-1} \tilde{X}^i \bar{\partial}^i, \\ B_n &= \sum_{i=0}^{2n-1} S^i \partial^i + \sum_{i=0}^{2n-1} \tilde{S}^i \bar{\partial}^i, \end{aligned}$$

here $X^i, \tilde{X}^j, S^i, \tilde{S}^j$ ($i, j = 0, \dots, 2n-1$) are 2×2 deformation matrices. The compatibility conditions (6) give the following deformation equations

$$\frac{\partial}{\partial t_n} p = \partial^{2n+1} p + \bar{\partial}^{2n+1} p + \dots,$$

which consist the mVN-hierarchy. The deformation of the eigenfunctions of the operator L are defined by the system

$$\frac{\partial}{\partial t_n} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = A_n \begin{pmatrix} \psi \\ \varphi \end{pmatrix}. \quad (7)$$

When $n = 1$ we have the first member of the mVN-hierarchy:

$$p_t = p_{t+} + p_{t-}, \quad (8)$$

where

$$\begin{aligned} p_{t+} &= p_{zzz} + 3p_z \omega + \frac{3}{2} p \omega_z, \\ p_{t-} &= p_{\bar{z}\bar{z}\bar{z}} + 3p_{\bar{z}} \bar{\omega} + \frac{3}{2} p \bar{\omega}_{\bar{z}}. \end{aligned}$$

Further, the deformation of the eigenfunctions ψ, φ via (7) generates the corresponding deformations of the coordinates $X(z, \bar{z}, t)$ of the surface conformally immersed into space $\mathbb{R}^{3,0}$.

It is easy to verify that the matrices of the group \tilde{G} , which double covers the group of the fractional linear transformations of the Riemann surface S ,

are coincide with matrices of a matrix representation of the quaternion algebra $\mathcal{O}_{0,2} = \mathbb{H}$. Indeed, a general element of \mathbb{H} has a form

$$\mathcal{A} = a^0 \mathbf{e}_0 + a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^{12} \mathbf{e}_{12},$$

where $a^0, a^1, a^2, a^{12} \in \mathbb{R}$. Further, in virtue of the maps $\mathbf{e}_i \rightarrow \sigma_i$ ($i = 0, 1, 2$), where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

we have

$$a^0 \sigma_0 + a^1 \sigma_1 + a^2 \sigma_2 + a^{12} \sigma_{12} = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix},$$

where

$$A = a^1 + ia^{12}, \quad B = a^2 + ia^2.$$

Therefore, $\mathbf{Spin}(2) \subset \tilde{G}$. Let $Q \times_{\tilde{G}} \Delta_2$ be a spinor bundle of the Riemann surface S , where Δ_2 is a spinor representation of the group \tilde{G} , and let (ψ, φ) are smooth sections of the bundle $Q \times_{\tilde{G}} \Delta_2$, where the spinors ψ, φ satisfy the Dirac equation (5). Further, let $\beta : Q \times_{\tilde{G}} \Delta_2 \longrightarrow Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}$ be a spinor bundle map between the spinor bundle of S and spinor bundle of the manifold $M^{2,0}$. Then in virtue of the map β we can to identify the sections (spinor fields) of these bundles

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix} \sim \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}.$$

In so doing, $\mathbf{e}_1 \rightarrow 1 \cdot x$, $\mathbf{e}_2 \rightarrow i \cdot y$, and $\mathbf{e}_1 \cdot \nabla_{\mathbf{e}_1}^{M^{2,0}} \rightarrow \frac{\partial}{\partial x}$, $\mathbf{e}_2 \cdot \nabla_{\mathbf{e}_2}^{M^{2,0}} \rightarrow i \frac{\partial}{\partial y}$. Therefore, for the Dirac operator of $M^{2,0}$ we have

$$D \sim \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Thus, in virtue of the spinor bundle map $\beta : Q \times_{\tilde{G}} \Delta_2 \longrightarrow Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}$ the systems (2) and (5) are equivalent. Therefore, if Φ is a parallel spinor field on the manifold $M^{3,0}$, then integrable deformations of the restriction $\Phi|_{M^{2,0}} = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$, where φ^+ and φ^- are the half-spinors (Weyl spinors) of the surface $M^{2,0}$, are defined by the mVN-hierarchy.

4 Surfaces of revolution

Let us consider now one important particular case, namely, the case when the surface $M^{2,0}$ is a surface of revolution. In this case [KT95, KT96, Tai97a, Tai97b] the functions ψ and φ take the form

$$\psi = r(x) \exp(\lambda y), \quad \varphi = s(x) \exp(\lambda y), \quad (9)$$

and the system (5)

$$\begin{aligned} \frac{\partial}{\partial z} \psi &= p\varphi, \\ \frac{\partial}{\partial \bar{z}} \varphi &= -p\psi, \end{aligned}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),$$

reduces to a system

$$\begin{aligned} \frac{\partial}{\partial x} r + i\lambda r &= us, \\ \frac{\partial}{\partial x} s - i\lambda s &= -ur, \end{aligned} \quad (10)$$

where $u = 2p$, and $p(x)$ is a real-valued function. The system (10) is nothing but a well-known *Zakharov-Shabat system* [ZS71]. Solutions of this system can be obtained via the Bargmann potentials [Bar49] (see also [Lam80]):

$$\begin{aligned} r &= e^{-i\lambda x} [4i\lambda + a(x)], \\ s &= e^{-i\lambda x} b(x). \end{aligned} \quad (11)$$

Indeed, substituting the Bargmann potentials (11) into (10) we find

$$a_x = ub, \quad b = 2u, \quad b_x = -ua. \quad (12)$$

Excluding u from the first and third equations (12) and then integrating it we obtain $a^2 + b^2 = 4\mu^2$, where μ is a constant of integration. Further, from the first and second equations (12) we have $b^2 = 2a_x$. Thus, the function $a(x)$ satisfies the equation

$$a_x + a^2/2 = 2\mu^2.$$

After substitution $a = 2w_x/w$ we came to the linear equation

$$w_{xx} - \mu^2 w = 0. \quad (13)$$

The solution of the equation (13) has a form

$$w = \alpha e^{\mu x} + \beta e^{-\mu x} = 2e^\phi \cosh(\mu x - \phi),$$

where $\phi = \frac{1}{2} \ln(\beta/\alpha)$. Then from the first and second equations (12) we obtain

$$u^2 = \frac{1}{2} a_x = \mu^2 \operatorname{sech}^2(\mu x - \phi)$$

and finally

$$u = \pm \mu \operatorname{sech}(\mu x - \phi). \quad (14)$$

Let $\mu > 0$, from (12) we find the functions $b(x)$ and $a(x)$:

$$\begin{aligned} b(x) &= \pm 2\mu \operatorname{sech}(\mu x - \phi), \\ a(x) &= 4\mu^2 \int \operatorname{sech}^2(\mu x - \phi) dx = 4\mu^2 \tanh(\mu x - \phi). \end{aligned}$$

Substituting now the functions $a(x), b(x)$ into (11) we find finally the fundamental solutions (Jost functions) of the system (10):

$$\begin{aligned} \phi_1^+(x, \lambda) &= e^{-i\lambda x} \frac{2i\lambda + \mu \tanh(\mu x - \phi)}{2i\lambda - \mu}, \\ \phi_2^+(x, \lambda) &= \pm e^{-i\lambda x} \frac{\operatorname{sech}(\mu x - \phi)}{2i\lambda - \mu}. \end{aligned} \quad (15)$$

Therefore, *in the case of surface of revolution the sections (ψ, φ) of the spinor bundle $Q \times_{\tilde{G}} \Delta_2$ are expressed via the fundamental solutions of the Zakharov-Shabat system as follows*

$$\begin{aligned} \psi &= \phi_1^+(x, \lambda) e^{\lambda y}, \\ \varphi &= \phi_2^+(x, \lambda) e^{\lambda y}. \end{aligned} \quad (16)$$

Thus, in this case we obtain an explicit form of the spinors ψ and φ . Moreover, in virtue of the spinor bundle map $Q \times_{\tilde{G}} \Delta_2 \longrightarrow Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}$ we can to express the half-spinors φ^+, φ^- of the surface $M^{2,0}$ via the solutions (15).

Further, the mVN-hierarchy related with the system (5) in the case of surface of revolution reduces to a modified Korteweg-de Vries hierarchy. Indeed, for the first member (8) of the mVN-hierarchy the each parts p_{t+} and p_{t-} equals to $p_{xxx} + 6p^2 p_x$. Take into account that $p = u/2$ we came to a famous modified Korteweg-de Vries equation

$$u_t = u_{xxx} + \frac{3}{2} u^2 u_x. \quad (17)$$

Therefore, integrable deformations of the surfaces of revolution and also a deformation of the spinor field (ψ, φ) are defined by the mKdV-hierarchy. Substitute now $u = \sqrt{4}v$ into (17) we obtain a standard form of the mKdV-equation,

$$v_t - v_{xxx} - 6v^2v_x = 0. \quad (18)$$

Further, plugging $v = u/\sqrt{4}$ into (18), where u is a potential of the form (14), we see that a dependence on t is expressed by an equation

$$\phi_t + \mu^3 = 0.$$

Hence it immediately follows that $u = \pm \operatorname{sech}(\mu x - \mu^3 t)$. Therefore, an explicit dependence of (ψ, φ) on deformation variable t is defined by the expressions

$$\begin{aligned} \psi &\sim \phi_1^+(x, \lambda, t)e^{\lambda y} = e^{\lambda(y-ix)} \frac{2i\lambda + \mu \tanh(\mu x - \mu^3 t)}{2i\lambda - \mu}, \\ \varphi &\sim \phi_2^+(x, \lambda, t)e^{\lambda y} = e^{\lambda(y-ix)} \frac{\operatorname{sech}(\mu x - \mu^3 t)}{2i\lambda - \mu}. \end{aligned} \quad (19)$$

Thus, *integrable deformations of the spinor field (ψ, φ) on the surface of revolution in the case of one-soliton solution of the Zakharov-Shabat system (10) defining by the linear Bargmann potentials (11), are expressed by the mKdV-equation and have an explicit form (19).*

Owing to absence of deformation, solutions (16) will be called *static solutions*.

Further generalization is a transition to multi-soliton solutions (for example, quadratic Bargmann potentials correspond to a two-soliton solution), and also consideration of higher members of the mKdV-hierarchy, which will be considered in a separate paper.

In conclusion, by virtue of the map $\beta : Q \times_{\tilde{G}} \longrightarrow Q \times_{\mathbf{Spin}(2,0)} \Delta_{2,0}$ and relation between smooth sections of these bundles, $(\psi, \varphi) \sim (\varphi^+, \varphi^-)$, follows that *if Φ is a parallel spinor field on the manifold $M^{3,0}$, and the surface $M^{2,0}$ conformally immersing into $M^{3,0}$ is homeomorphic to a surface of revolution, then integrable deformations of the restriction $\Phi|_{M^{2,0}} = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$, where φ^+ and φ^- are the Weyl spinors of the surface $M^{2,0}$, are defined by the modified Korteweg-de Vries hierarchy.*

References

- [Abr89] U. Abresch, *Spinor representation of CMC surfaces*, Lecture at Luminy (1989).
- [AAF98] I. Agricola, B. Ammann, Th. Friedrich, *A comparison of the eigenvalues of the Dirac and Laplace operator on the two-dimensional torus*, preprint SFB 288, N 361, TU-Berlin (1998).

- [AF98] I. Agricola, Th. Friedrich, *Upper bounds for the first eigenvalue of the Dirac operator on surfaces*, preprint SFB 288, N 330, TU-Berlin (1998); J. Geometry and Physics (to appear).
- [Amm98] B. Ammann, **Spin-Strukturen und das Spektrum des Dirac Operators** (Dissertation Freiburg 1998, Shaker-Verlag, Aachen 1998).
- [At71] M.F. Atiyah, *Riemann surfaces and spin structures*, Ann. Scient. Ecole Norm. Sup. **4**, 47-62 (1971).
- [Bar49] V. Bargmann, *On the connection between phase shifts and scattering potentials*, Rev. Mod. Phys. **21**, 488-493 (1949).
- [BP88] A.O. Barut, M. Pavšič, *The spinning minimal surfaces without the Grassmann variables*, Lett. Math. Phys. **16**, 333-338 (1988).
- [Bär91] C. Bär, **Das Spektrum von Dirac-Operatoren** (Dissertation, Bonner Math. Schriften **217**, 1991).
- [Bau81] H. Baum, **Spin-Strukturen und Dirac-Operatoren über pseudo-Riemannschen Mannigfaltigkeiten** (Leipzig, Teubner-Verlag, 1981).
- [BFGK] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, **Twistors and Killing spinors on Riemannian manifolds** (Teubner-Verlag Leipzig/Stuttgart 1991).
- [Bob94] A.I. Bobenko, *Surfaces in terms of 2×2 matrices. Old and new integrable cases*, in **Harmonic maps and integrable systems** (A. Fordy, J. Wood eds.), 83-127, Vieweg (1994).
- [Bud86] P. Budinich, *Null vectors, spinors and strings*, Commun. Math. Phys. **107**, 455-465 (1986).
- [BDF86] P. Budinich, L. Dabrowski, P. Furlan, *Minimal surfaces and strings from spinors: a realization of the Cartan programme*, Nuovo Cim. **A96**, 194-211 (1986).
- [BR87] P. Budinich, M. Rigoli, *Cartan spinors, minimal surfaces, and strings*, SISSA (Trieste) preprint 95/87/F.M. (1987).
- [CK96] R. Carroll, B.G. Konopelchenko, *Generalized Weierstrass-Enneper inducing, conformal immersion and gravity*, Int. J. Modern Physics A **11**, 1183-1216 (1996).
- [Che54] C. Chevalley, **The Algebraic Theory of Spinors** (Columbia University Press, New York, 1954).

- [Eisen] L. P. Eisenhart, **A treatise on the differential geometry of Curves and Surfaces** (Dover, New York, 1909).
- [Fr97a] Th. Friedrich, **Dirac-Operatoren in der Riemannschen Geometrie** (Vieweg-Verlag Braunschweig/Wiesbaden, 1997).
- [Fr97b] Th. Friedrich, *On the Spinor Representation of Surfaces in Euclidean 3-Space*, preprint SFB 288, N 295, TU-Berlin,(1997); J. Geometry and Physics (to appear).
- [HO83] D.A. Hoffman, R. Osserman, *The Gauss map of surfaces in R^n* , J. Differential Geometry **18**, 733-754 (1983).
- [HO85] D.A. Hoffman, R. Osserman, *The Gauss map of surfaces in R^3 and R^4* , Proc. London Math. Soc. **50**, 27-56 (1985).
- [John80] D. Johnson, *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. **22**, 365-373 (1980).
- [Ken79] K. Kenmotsu, *Weierstrass formula for surfaces of prescribed mean curvature*, Math. Ann. **245**, 89-99 (1979).
- [Kon96] B.G. Konopelchenko, *Induced surfaces and their integrable dynamics*, Stud. Appl. Math. **96**, 9-51, (1996).
- [Kon98] B.G. Konopelchenko, *Weierstrass representation for surfaces in 4D spaces and their integrable deformations via the DS hierarchy*, preprint math.DG/9807129, (1998).
- [KL97] B.G. Konopelchenko, G. Landolfi, *On classical string configuration*, Mod. Phys. Lett. **A12**, 3161-3168 (1997).
- [KL98a] B.G. Konopelchenko, G. Landolfi, *Generalized Weierstrass representation for surfaces in multidimensional Riemann spaces*, preprint math.DG/9804144, (1998); J. Geometry and Physics (to appear).
- [KL98b] B.G. Konopelchenko, G. Landolfi, *Induced surfaces and their integrable dynamics. II. Generalized Weierstrass representation in 4D spaces and deformations via DS hierarchy*, preprint math.DG/9810138, (1998).
- [KL98c] B.G. Konopelchenko, G. Landolfi, *Quantum effects for extrinsic geometry of strings via the generalized Weierstrass representation*, preprint hep-th/9810209 (1998).
- [KL99] B.G. Konopelchenko, G. Landolfi, *On rigid string instantons in four dimensions*, preprint hep-th/9901113 (1999).

- [KT95] B.G. Konopelchenko, I.A. Taimanov, *Generalized Weierstrass formulae, soliton equations and Willmore surfaces*, preprint N 187, Univ. Bochum (1995).
- [KT96] B.G. Konopelchenko, I.A. Taimanov, *Constant mean curvature surfaces via an integrable dynamical system*, J. Phys. A: Math. Gen. **29**, 1261-1265 (1996).
- [KS96] R. Kusner, N. Schmitt, *The spinor representation of surfaces in space*, preprint dg-ga/9610005, (1996).
- [Lam80] G.L. Lamb, Jr., **Elements of soliton theory** (Jhon Wiley & Sons, New York 1980).
- [Lou81] P. Lounesto, *Scalar Products of Spinors and an Extension of Brauer-Wall Groups*, Found. Phys. **11**, 721-740 (1981).
- [LM89] H.B. Lawson, M.-L. Michelsohn, **Spin Geometry** (Princeton University Press, Princeton 1989).
- [Mat97a] S. Matsutani, *Constant mean curvature surfaces and the Dirac operator*, J. Phys. A: Math. Gen. **30**, 4019-4024 (1997).
- [Mat97b] S. Matsutani, *Immersion anomaly of Dirac operator for surfaces in \mathbb{R}^3* , preprint physics/9707003 (1997).
- [Mat98] S. Matsutani, *Dirac operators of a conformal surface immersed in \mathbb{R}^4 : further generalized Weierstrass relation*, preprint solv-int/9801006 (1998).
- [Port69] I.R. Porteous, **Topological Geometry** (van Nostrand, London, 1969).
- [Roz55] B.A. Rozenfel'd, **Neevklidovy geometrii [Non-Euclidean Geometries]** (Moscow, 1955) [in Russian].
- [Sul89] D. Sullivan, *The spinor representation of minimal surfaces*, Notes (1989).
- [Sym85] A. Sym, *Soliton surfaces and their application* in: Soliton geometry from spectral problems, Lecture Notes in Physics 239, Springer, Berlin 1985, 154-231.
- [Tai97a] I.A. Taimanov, *Modified Novikov-Veselov equation and differential geometry of surfaces*, Trans. Amer. Math. Soc. Ser.2 **179**, 133-159 (1997).
- [Tai97b] I.A. Taimanov, *Surfaces of revolution in terms of solitons*, Ann. Glob. Anal. Geom. **15**, 419-435 (1997).

- [Tai97c] I.A. Taimanov, *The Weierstrass representation of closed surfaces in \mathbb{R}^3* , preprint SFB 288, N 291, TU-Berlin (1997).
- [Tai98] I.A. Taimanov, *The Weierstrass representation of spheres in \mathbb{R}^3 , the Willmore numbers and soliton spheres*, preprint SFB 288, N 302, TU-Berlin (1998).
- [Var98a] V.V. Varlamov, *Generalized Weierstrass representation for surfaces in terms of Dirac-Hestenes spinor field*, preprint math.DG/9807152; J. Geometry and Physics (to appear).
- [Var98b] V.V. Varlamov, *Dirac-Hestenes spinors and Weierstrass representation for surfaces in 4D complex space*, preprint math.DG/9812099 (1998).
- [VP94] K.S. Viswanathan, R. Parthasarathy, *A conformal field theory of extrinsic geometry of 2-d surfaces*, preprint hep-th/9411042 (1994).
- [Weier] K. Weierstrass, *Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist*, Monatsber. Acad. Wiss. Berlin, 1866, s.612-625.
- [Yam99] K. Yamagishi, *A Note on modified Veselov-Novikov Hierarchy*, preprint math-ph/9904017 (1999); Physics Lett. **B** (to appear).
- [ZS71] V.E. Zakharov, A.B. Shabat, *Exact theory of two-focussing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys.-JETP **61**, 118-134 (1971).